

THE LARGEST EIGENVALUE AND BI-AVERAGE DEGREE OF A GRAPH

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ABSTRACT. We show that for a graph G with the vertex set V and the largest eigenvalue $\lambda_{\max}(G)$, letting

$$M(G) := \max_{\emptyset \neq X, Y \subseteq V} \frac{e(X, Y)}{\sqrt{|X||Y|}}$$

(where $e(X, Y)$ denotes the number of edges between X and Y), we have

$$M(G) \leq \lambda_{\max}(G) \leq \left(\frac{1}{4} \log |V| + 1\right) M(G).$$

Here the lower bound is attained if G is regular or bi-regular, whereas the logarithmic factor in the upper bound, conjecturally, can be improved — although we present an example showing that it cannot be replaced with a factor growing slower than $(\log |V| / \log \log |V|)^{1/8}$.

Further refinements are established, particularly in the case where G is bipartite.

1. BACKGROUND AND SUMMARY OF RESULTS: THE GENERAL CASE

For a graph G , by $\lambda_{\max}(G)$ we denote the largest eigenvalue of G (often referred to as the *first* eigenvalue), by $\bar{d}(G)$ the average degree of G , and by $\Delta(G)$ the maximum degree of G . All graphs throughout are simple.

It is well-known and easy to prove that if d is the degree sequence of a graph G , then, denoting by $\|d\|_2$ the ℓ^2 -norm of d , we have

$$\|d\|_2 \leq \lambda_{\max}(G) \leq \Delta(G); \tag{1}$$

in particular, if G is r -regular, then $\lambda_{\max}(G) = r$. This basic observation determines completely the meaning of the largest eigenvalue for regular graphs and indeed, for all graphs which are “nearly regular” in the sense that, say, the maximum degree does not exceed a constant multiple of the average degree.

In this paper we investigate the general case, where a significant gap between the ℓ^2 -norm of the degree sequence and the maximum degree can potentially exist, with the ultimate goal to understand the relation between the largest eigenvalue of a graph and its degree sequence in this case.

We notice that one can expect the largest eigenvalue to reflect mostly the properties of the degree sequence, and to a much lesser extent the structure of the graph itself, as there exist graphs with the same number of vertices and the same largest eigenvalue which do

not look similar — as, for instance, all graphs $K_{n_1, n_2} \cup \overline{K}_{n-(n_1+n_2)}$ with the product $n_1 n_2$ and integer $n > n_1 + n_2$ fixed.

The last example (more generally, the fact that the largest eigenvalue is monotonic by the interlacing theorem, while the degree sequence is easy to manipulate, say, by adding isolated vertices), suggests that it is insufficient for our purposes to confine to the graph itself. Instead, we have to bring into consideration the whole family of its induced subgraphs; more precisely, of the induced subgraphs of its bipartite double cover. Specifically, suppose that G is a graph on the vertex set V , and let $X, Y \subseteq V$ be non-empty sets of vertices. Consider the subgraph $G_{X,Y}$ of the bipartite double cover $G \times K_2$, induced by X and Y (or rather copies thereof, taken in different partite sets). Thus, $G_{X,Y}$ is a bipartite graph with disjoint copies of X and Y as the partite sets, and with the number of edges equal to the number of edges between X and Y in G . Using the standard notation $e(X, Y)$ for this number of edges, the average degree of a vertex from X in $G_{X,Y}$ is $e(X, Y)/|X|$, and the average degree in $G_{X,Y}$ of a vertex from Y is $e(X, Y)/|Y|$. The geometric mean of these averages, which is $e(X, Y)/\sqrt{|X||Y|}$, can thus be considered as a measure of the average degree of $G_{X,Y}$. We give this measure a designated name, defining the *bi-average degree* of a bipartite graph with the partite sets U and W to be $e(U, W)/\sqrt{|U||W|}$. Therefore, the quantity

$$\mathbf{M}(G) := \max_{\emptyset \neq X, Y \subseteq V} \frac{e(X, Y)}{\sqrt{|X||Y|}}$$

can be interpreted as the maximum bi-average degree of an induced subgraph of the bipartite double cover of G .

We notice that for any graph G we have

$$\overline{d}(G) \leq \mathbf{M}(G) \leq \Delta(G), \quad (2)$$

the lower bound following from the fact that, denoting by V the vertex set of G , the bi-average degree of the whole bipartite double cover $G \times K_2$ is $\frac{e(V, V)}{|V|} = \overline{d}(G)$, and the upper bound from

$$e(X, Y) \leq \min \{|X| \Delta(G), |Y| \Delta(G)\} \leq \Delta(G) \sqrt{|X||Y|}, \quad X, Y \subseteq V.$$

As a consequence of (2), if G is r -regular, then $\mathbf{M}(G) = r$.

With this notation, we can state our main results (to be proved in subsequent sections).

Theorem 1. *If G is a graph with the vertex set V , then*

$$\mathbf{M}(G) \leq \lambda_{\max}(G) \leq \left(\frac{1}{4} \log |V| + 1\right) \mathbf{M}(G).$$

By the remark above, and since $\lambda_{\max}(G) = r$ for an r -regular graph G , the lower bound of Theorem 1 is attained if G is regular. As to the upper bound, we have no reasons to

believe that it is sharp. However, in Section 5 we construct a sequence of graphs G of arbitrarily large order n such that

$$\mathbf{M}(G) \ll \left(\frac{\log \log n}{\log n} \right)^{1/8} \lambda_{\max}(G)$$

(with an absolute implicit constant); this shows that the upper bound of Theorem 1 cannot be improved all the way down to the lower bound. Our construction uses a version of the tensor power trick and its analysis is rather complicated technically; finding a simpler construction would be interesting.

We notice that the close relation between the quantity $\mathbf{M}(G)$ and the largest eigenvalue $\lambda_{\max}(G)$ stems from the fact that if A denotes the adjacency matrix of G , and n is the order of G , then

$$\mathbf{M}(G) = \max_{0 \neq \xi, \eta \in \{0,1\}^n} \frac{\xi^t A \eta}{\|\xi\| \|\eta\|} \quad (3)$$

(as it follows by associating to every subset of the vertex set of G its characteristic vector), whereas

$$\lambda_{\max}(G) = \sup_{0 \neq x, y \in \mathbb{R}^n} \frac{x^t A y}{\|x\| \|y\|}.$$

This observation immediately yields the estimate

$$\mathbf{M}(G) \leq \lambda_{\max}(G),$$

which was included into the statement of Theorem 1 (and will be included also into subsequent theorems) just for completeness.

Our next result presents an improvement over Theorem 1 for sparse graphs.

Theorem 2. *If G is a graph with the vertex set V , then*

$$\mathbf{M}(G) \leq \lambda_{\max}(G) \leq \sqrt{(\log \Delta(G) + 1) \left(\frac{1}{4} \log |V| + 1 \right)} \mathbf{M}(G).$$

To present a yet more robust estimate, we introduce the following notation. Given a finite sequence d with non-negative terms, consider the non-increasing rearrangement $d_1 \geq \dots \geq d_n \geq 0$ of the terms of d , define $k \in [1, n]$ to be the smallest positive integer with

$$d_1^2 + \dots + d_k^2 \geq d_{k+1}^2 + \dots + d_n^2,$$

and let $\rho(d) := d_1/d_k$ if $d_k \neq 0$, and $\rho(d) := 1$ if $d_k = 0$ (in which case $k = 1$ and d is the zero sequence). The quantity $\rho(d)$ measures how smooth is d . We record the following simple bounds:

- i) $\rho(d) \geq 1$. (Equality is attained, for instance, if all positive coordinates of d are equal to each other.)

ii) $\rho(d) \leq \sqrt{n}$: for $k = 1$ this is trivial, and for $k > 1$ follows from

$$d_1^2 \leq d_1^2 + \cdots + d_{k-1}^2 < d_k^2 + \cdots + d_n^2 < nd_k^2.$$

(On the other hand, if $d_1 = \sqrt{n}$ and $d_2 = \cdots = d_n = 1 + 1/n$, then $\rho(d) = (1 - o(1))\sqrt{n}$ as $n \rightarrow \infty$.)

iii) $\rho(d) < \sqrt{2} \|d\|_\infty / \|d\|_2$ (with $\|\cdot\|_p$ denoting the ℓ^p -norm): to see this, notice that $\|d\|_\infty = d_1$ and

$$nd_k^2 \geq d_k^2 + \cdots + d_n^2 > \frac{n}{2} \|d\|_2^2.$$

Theorem 3. *Let G be a graph with the vertex set V . For each $v \in V$ and $X \subseteq V$, denote by $d_X(v)$ the number of neighbors of v in X , and let $K := \max_{\emptyset \neq X \subseteq V} \rho((d_X(v))_{v \in V})$. Then*

$$\mathbf{M}(G) \leq \lambda_{\max}(G) \leq \sqrt{2(\log K + 1)(\log |V| + 4)} \mathbf{M}(G).$$

Clearly, for any sequence d with integer terms we have $\rho(d) \leq \|d\|_\infty$. Consequently, Theorem 3 readily implies Theorem 2, albeit with slightly weaker constants.

2. BACKGROUND AND SUMMARY OF RESULTS: BIPARTITE GRAPHS

Theorems 1–3 can be refined in the situation where the graph G under consideration is bipartite. Indeed, the very definition of the quantity $\mathbf{M}(G)$ can be given a cleaner shape in this case.

Claim 1. *If G is a bipartite graph with the partite sets U and W , then*

$$\mathbf{M}(G) = \max_{\substack{\emptyset \neq X \subseteq U \\ \emptyset \neq Y \subseteq W}} \frac{e(X, Y)}{\sqrt{|X||Y|}}.$$

Proof. It suffices to show that for any $\emptyset \neq X_U, Y_U \subseteq U$ and $\emptyset \neq X_W, Y_W \subseteq W$ we have

$$\frac{e(X_U \cup X_W, Y_U \cup Y_W)}{\sqrt{(|X_U| + |X_W|)(|Y_U| + |Y_W|)}} \leq \max \left\{ \frac{e(X_U, Y_W)}{\sqrt{|X_U||Y_W|}}, \frac{e(X_W, Y_U)}{\sqrt{|X_W||Y_U|}} \right\}.$$

To this end we denote by T the maximum in the right-hand side, so that $e(X_U, Y_W) \leq T\sqrt{|X_U||Y_W|}$ and $e(X_W, Y_U) \leq T\sqrt{|X_W||Y_U|}$, and observe that then

$$\begin{aligned} e(X_U \cup X_W, Y_U \cup Y_W) &= e(X_U, Y_W) + e(X_W, Y_U) \\ &\leq T(\sqrt{|X_U||Y_W|} + \sqrt{|X_W||Y_U|}) \\ &\leq T\sqrt{(|X_U| + |X_W|)(|Y_U| + |Y_W|)}. \end{aligned}$$

□

The following corollary will be used in conjunction with the fact that the largest eigenvalue of the bipartite double cover of a graph is equal to the largest eigenvalue of the graph itself.

Corollary 1. *For any graph G we have $M(G \times K_2) = M(G)$.*

To prove the corollary denote by φ' and φ'' the adjacency-preserving bijections of the vertex set V of G onto the partite sets of $G \times K_2$, and notice that then, for any $X, Y \subseteq V$,

$$\frac{e_G(X, Y)}{\sqrt{|X||Y|}} = \frac{e_{G \times K_2}(\varphi'(X), \varphi''(Y))}{\sqrt{|\varphi'(X)||\varphi''(Y)|}}$$

(where e_G and $e_{G \times K_2}$ denote the number of edges in the corresponding graphs).

The bipartite analogue of (1) is given by

Lemma 1. *If G is a bipartite graph with the partite sets U and W then, denoting by d_U and d_W the degree sequences of U and W , respectively, and letting $\Delta_U := \|d_U\|_\infty$ and $\Delta_W := \|d_W\|_\infty$, we have*

$$\max \left\{ \sqrt{\frac{|U|}{|W|}} \|d_U\|_2, \sqrt{\frac{|W|}{|U|}} \|d_W\|_2 \right\} \leq \lambda_{\max}(G) \leq \sqrt{\Delta_U \Delta_W}.$$

Consequently,

$$\sqrt{\|d_U\|_2 \|d_W\|_2} \leq \lambda_{\max}(G) \leq \sqrt{\Delta_U \Delta_W},$$

and, therefore, if G is (r_U, r_W) -regular, then $\lambda_{\max} = \sqrt{r_U r_W}$.

Proof. Let A denote the adjacency matrix of G . If $\xi \in \mathbb{R}^{|U|+|W|}$ is the characteristic vector of U , then the non-zero coordinates of the vector $A\xi$ form the sequence d_W , and therefore $\|A\xi\| = \sqrt{|W|} \|d_W\|_2$. Hence,

$$\lambda_{\max}(G) = \sup_{0 \neq x \in \mathbb{R}^{|U|+|W|}} \frac{\|Ax\|}{\|x\|} \geq \frac{\|A\xi\|}{\|\xi\|} = \sqrt{\frac{|W|}{|U|}} \|d_W\|_2,$$

and in an identical way we obtain the estimate $\lambda_{\max}(G) \geq \sqrt{|U|/|W|} \|d_U\|_2$.

For the upper bound, suppose that $(\xi_u, \eta_w)_{u \in U, w \in W}$ is an eigenvector of A , corresponding to the eigenvalue $\lambda_{\max}(G)$; thus,

$$\sum_{w \sim u} \eta_w = \lambda_{\max}(G) \xi_u \quad \text{and} \quad \sum_{u \sim w} \xi_u = \lambda_{\max}(G) \eta_w, \quad (4)$$

with the summation in the first sum extending over all vertices $w \in W$ adjacent to the given vertex $u \in U$, and the summation in the second sum over all vertices $u \in U$ adjacent to the given vertex $w \in W$. Letting

$$\xi_{\max} := \max_{u \in U} |\xi_u| \quad \text{and} \quad \eta_{\max} := \max_{w \in W} |\eta_w|,$$

we conclude that

$$\lambda_{\max}(G) \xi_{\max} \leq \Delta_U \eta_{\max} \quad \text{and} \quad \lambda_{\max}(G) \eta_{\max} \leq \Delta_W \xi_{\max},$$

and the result follows by multiplying out the two estimates and observing that $\xi_{\max} \eta_{\max} \neq 0$. (If we had, say, $\xi_{\max} = 0$, this would imply $\xi_u = 0$ for each $u \in U$ and, consequently, $\eta_w = 0$ for each $w \in W$ by (4).) \square

The bipartite analogue of (2) is as follows: if $G, U, W, d_U, d_W, \Delta_U$, and Δ_W are as in Lemma 1, then, letting $\bar{d}_U := \|d_U\|_1$ and $\bar{d}_W := \|d_W\|_1$, we have

$$\sqrt{\bar{d}_U \bar{d}_W} \leq M(G) \leq \sqrt{\Delta_U \Delta_W}. \quad (5)$$

For the proof it suffices to notice that, on the one hand,

$$M(G) \geq \frac{e(U, W)}{\sqrt{|U||W|}} = \sqrt{\frac{|U|}{|W|}} \bar{d}_U = \sqrt{\frac{|W|}{|U|}} \bar{d}_W,$$

and, on the other hand, for any $X \subseteq U$ and $Y \subseteq W$,

$$e(X, Y) \leq \min\{|X|\Delta_U, |Y|\Delta_W\} \leq \sqrt{|X||Y|} \sqrt{\Delta_U \Delta_W}.$$

Notice that, as a result of (5) and Lemma 1, for an (r_U, r_W) -regular graph G we have

$$M(G) = \lambda_{\max}(G) = \sqrt{r_U r_W}.$$

We now state the bipartite versions of Theorems 1–3.

Theorem 1'. *If G is a bipartite graph with the partite sets U and W , then*

$$M(G) \leq \lambda_{\max}(G) \leq \sqrt{\left(\frac{1}{4} \log |U| + 1\right) \left(\frac{1}{4} \log |W| + 1\right)} M(G).$$

Theorem 2'. *If G is a bipartite graph with the partite sets U and W , then, denoting by Δ_U the maximum degree of a vertex from U , we have*

$$M(G) \leq \lambda_{\max}(G) \leq \sqrt{(\log \Delta_U + 1) \left(\frac{1}{4} \log |W| + 1\right)} M(G).$$

Observing that in a bipartite graph the degree of a vertex from one partite set does not exceed the size of another partite set, we get the following corollary (to be compared with Theorem 1').

Corollary 2. *If G is a bipartite graph with the partite sets U and W , then, letting $n := \min\{|U|, |W|\}$, we have*

$$M(G) \leq \lambda_{\max}(G) \leq \left(\frac{1}{2} \log n + 2\right) M(G).$$

Theorem 3'. *Let G be a bipartite graph with the partite sets U and W . For each $u \in U$ and $Y \subseteq W$, denote by $d_Y(u)$ the number of neighbors of u in Y , and let $K := \max_{\emptyset \neq Y \subseteq W} \rho((d_Y(u))_{u \in U})$. Then*

$$\mathbf{M}(G) \leq \lambda_{\max}(G) \leq \sqrt{2(\log K + 1)(\log |W| + 4)} \mathbf{M}(G).$$

Theorems 1–3 follow immediately from Theorems 1'–3' using the following simple scheme: given a graph G , apply the appropriate theorem about bipartite graphs to the bipartite double cover $G \times K_2$, and then use Corollary 1 along with the fact that $\lambda_{\max}(G) = \lambda_{\max}(G \times K_2)$ to return back to the original graph G . For this reason, from now on we concentrate exclusively on the proofs of Theorems 1'–3'. In the next section we state three lemmas needed for the proofs, and deduce the theorems from the lemmas. The lemmas are proved in Section 4. In Section 5 we give an example which sets the limit to potentially possible improvements in Theorems 1–3'; namely, we construct graphs G of arbitrarily large order n such that

$$\lambda_{\max}(G) \gg \left(\frac{\log n}{\log \log n} \right)^{1/8} \mathbf{M}(G) \quad (6)$$

(with an absolute implicit constant).

3. AUXILIARY LEMMAS AND DEDUCTION OF THEOREMS 1'–3'

The three lemmas stated below in this section show that no vector is “almost orthogonal” simultaneously to all vertices of the unit cube $\{0, 1\}^n$; equivalently, there is no hyperplane to which all vertices of the unit cube are close simultaneously. Albeit slightly technical, these three lemmas are in the heart of our argument. Once the lemmas are stated, we show how Theorems 1'–3' follow from them. The lemmas themselves are proved in the next section.

By $\|\cdot\|$ we denote the usual Euclidean norm on a finite-dimensional real vector space. The standard inner product is denoted by $\langle \cdot, \cdot \rangle$. Thus, for instance, for an integer $n \geq 1$ and a vector $x \in \mathbb{R}^n$, we have $\langle x, x \rangle = \|x\|^2 = n\|x\|_2^2$.

Lemma 2. *Let $n \geq 1$ be an integer. For any vector $z \in \mathbb{R}^n$ with non-negative coordinates, there exists a non-zero vector $\delta \in \{0, 1\}^n$ such that*

$$\langle z, \delta \rangle \geq \frac{2}{\sqrt{\log n + 4}} \|z\| \|\delta\|.$$

Notice that the estimate of Lemma 2 is tight for $n = 1$. For a less trivial example, consider the vector $z := (1, 1/\sqrt{2}, \dots, 1/\sqrt{n})$, and notice that for any non-zero $\delta \in \{0, 1\}^n$ one has $\langle z, \delta \rangle < (2/\sqrt{\log n}) \|z\| \|\delta\|$.

Lemma 3. *Let $n, \Delta \geq 1$ be integers. For any integer vector $z \in [0, \Delta]^n$ there exists a non-zero vector $\delta \in \{0, 1\}^n$ such that*

$$\langle z, \delta \rangle \geq \frac{1}{\sqrt{\log \Delta + 1}} \|z\| \|\delta\|.$$

For our next lemma the reader may need to recall the definition of the function ρ introduced immediately after the statement of Theorem 2.

Lemma 4. *Let $n \geq 1$ be an integer. For any vector $z \in \mathbb{R}^n$ with non-negative coordinates, there exists a non-zero vector $\delta \in \{0, 1\}^n$ such that*

$$\langle z, \delta \rangle \geq \frac{1}{\sqrt{8(\log \rho(z) + 1)}} \|z\| \|\delta\|.$$

For a real matrix A , by $\|A\|$ we denote the operator norm of A ; that is,

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|},$$

with the Euclidean norms in the numerator and the denominator in the right-hand side. We recall that the operator norm of a symmetric matrix is equal to its largest eigenvalue, and that if A is a block matrix of the form $\begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix}$, then $\|A\| = \|B\| = \|B^t\|$. As a result, if G is a bipartite graph with the biadjacency matrix B , then $\lambda_{\max}(G) = \|B\| = \|B^t\|$.

We now deduce Theorems 1'–3' from Lemmas 2–4.

Proof of Theorem 1'. Write $m := |U|$ and $n := |W|$ and let B denote the biadjacency matrix of G , with rows corresponding to the elements of U , and columns to the elements of W . Fix $x \in \mathbb{R}^m \setminus \{0\}$ with $\|B^t x\| = \|B\| \|x\|$. Since all entries of B are non-negative, we can assume that all coordinates of x are non-negative. (If x have both positive and negative coordinates, then switching the signs of all negative coordinates yields a vector $x' \in \mathbb{R}^m \setminus \{0\}$ with $\|B^t x'\|/\|x'\| \geq \|B^t x\|/\|x\|$.) Hence, all coordinates of the vector $B^t x \in \mathbb{R}^n$ are non-negative, too, and applying Lemma 2 to this vector, we find a non-zero vector $\eta \in \{0, 1\}^n$ so that

$$\langle B^t x, \eta \rangle \geq \frac{2}{\sqrt{\log n + 4}} \|B^t x\| \|\eta\|.$$

Since $\langle B^t x, \eta \rangle = \langle x, B\eta \rangle \leq \|x\| \|B\eta\|$ and $\|B^t x\| = \|B\| \|x\|$, this gives

$$\|B\eta\| \geq \frac{2}{\sqrt{\log n + 4}} \|B\| \|\eta\|. \quad (7)$$

Applying now Lemma 2 to the vector $B\eta \in \mathbb{R}^m$, we find a non-zero vector $\xi \in \{0, 1\}^m$ with

$$\langle B\eta, \xi \rangle \geq \frac{2}{\sqrt{\log m + 4}} \|B\eta\| \|\xi\|.$$

Combining this with (7), we get

$$\xi^t B\eta = \langle B\eta, \xi \rangle \geq \frac{4}{\sqrt{(\log m + 4)(\log n + 4)}} \|B\| \|\xi\| \|\eta\|.$$

To complete the proof we notice that if $X \subseteq U$ is the subset with the characteristic vector ξ , and $Y \subseteq W$ is the subset with the characteristic vector η , then $|X| = \|\xi\|^2$, $|Y| = \|\eta\|^2$, and $e(X, Y) = \xi^t B\eta = \langle \xi, B\eta \rangle$, whence

$$\mathbf{M}(G) \geq \frac{e(X, Y)}{\sqrt{|X||Y|}} = \frac{\xi^t B\eta}{\|\xi\| \|\eta\|} \geq \frac{4}{\sqrt{(\log m + 4)(\log n + 4)}} \|B\|.$$

The result now follows in view of $\|B\| = \lambda_{\max}(G)$. \square

Proof of Theorem 2. We act as in the proof of Theorem 1', except that the second application of Lemma 2 is replaced with an application of Lemma 3. Specifically, let m, n, B, x , and η be as in the proof of Theorem 1', so that (7) holds true. Applying Lemma 3 to the vector $B\eta \in [0, \Delta_U]^m$, we find a non-zero vector $\xi \in \{0, 1\}^m$ with

$$\langle B\eta, \xi \rangle \geq \frac{1}{\sqrt{\log \Delta_U + 1}} \|B\eta\| \|\xi\|.$$

Comparing with (7) we obtain

$$\xi^t B\eta \geq \frac{2}{\sqrt{(\log \Delta_U + 1)(\log n + 4)}} \|B\| \|\xi\| \|\eta\|$$

and the rest of the argument is exactly as in the proof of Theorem 1'. \square

Proof of Theorem 3. We define m, n, B, x and η as in the proofs of Theorems 1' and 2', and this time replace the second application of Lemma 2 in Theorem 1' with an application of Lemma 4 to the vector $B\eta$, to find $\xi \in \{0, 1\}^m$ such that

$$\langle B\eta, \xi \rangle \geq \frac{1}{\sqrt{8(\log K + 1)}} \|B\eta\| \|\xi\|.$$

The proof then can be completed as those of Theorems 1' and 2'. \square

An important (though somewhat implicit) ingredient of the proofs of Theorems 1'–3' is the assertion that for any matrix B with non-negative entries, denoting by n the number of columns of B , we can find a non-zero vector $\eta \in \{0, 1\}^n$ satisfying (7). We notice that the coefficient in the right-hand side of (7) is essentially best possible, as one can easily check taking B to be the matrix of the orthogonal projection of \mathbb{R}^n onto the vector $(1, 1/\sqrt{2}, \dots, 1/\sqrt{n})$. It is quite possible, however, that this coefficient can be improved

in the special case where the entries of B are restricted to the values 0 and 1. A result of this sort would immediately lead to an improvement in Theorems 1–3'.

4. PROOFS OF LEMMAS 2–4

Proof of Lemma 2. We write $z = (z_1, \dots, z_n)$ and, without loss of generality, assume that

$$z_1 \geq \dots \geq z_n \geq 0 \quad \text{and} \quad \|z\| = 1. \quad (8)$$

Let $\tau := 2/\sqrt{\log n + 4}$. We will show that there exists $k \in [n]$ with $z_1 + \dots + z_k \geq \tau\sqrt{k}$; choosing then δ to be the vector with the first k coordinates equal to 1 and the rest equal to 0 completes the proof.

Suppose, for a contradiction, that $z_1 + \dots + z_k < \tau\sqrt{k}$ for $k = 1, \dots, n$. Multiplying this inequality by $z_k - z_{k+1}$ for each $k \in [n-1]$, and by z_n for $k = n$, adding up the resulting estimates, and rearranging the terms, we obtain

$$z_1^2 + \dots + z_n^2 < \tau(z_1 + (\sqrt{2} - 1)z_2 + \dots + (\sqrt{n} - \sqrt{n-1})z_n).$$

Using Cauchy-Schwartz and recalling (8) gives

$$1 < \tau \left(\sum_{k=1}^n (\sqrt{k} - \sqrt{k-1})^2 \right)^{1/2} \leq \frac{1}{2} \tau \sqrt{\log n + 4}$$

(we omit the routine estimate of the last sum), a contradiction. \square

Proof of Lemma 3. For every $i \in [0, \Delta]$, let n_i denote the number of coordinates of z which are equal to i , so that $n = n_0 + n_1 + \dots + n_\Delta$ and $\|z\|^2 = n_1 + \dots + \Delta^2 n_\Delta$. Consider the vector $\delta_i \in \{0, 1\}^n$ with each coordinate being 1 whenever the corresponding coordinate of z is at least i , and being 0 otherwise. We have $\|\delta_i\|^2 = n_i + \dots + n_\Delta$ and, as a result,

$$\langle \delta_i, z \rangle = in_i + \dots + \Delta n_\Delta \geq i \|\delta_i\|^2.$$

Consequently, if

$$\langle \delta_i, z \rangle < \tau \|z\| \|\delta_i\|$$

holds for some $\tau > 0$ and every $i \in [1, \Delta]$, then

$$\langle \delta_i, z \rangle^2 < \tau^2 \|z\|^2 \cdot \frac{1}{i} \langle \delta_i, z \rangle,$$

implying

$$i(in_i + \dots + \Delta n_\Delta) = i \langle \delta_i, z \rangle < \tau^2 \|z\|^2, \quad i \in [1, \Delta].$$

Dividing through by i and taking the sum over all $i \in [1, \Delta]$ yields

$$\tau^2 \|z\|^2 (\log \Delta + 1) > \sum_{i=1}^{\Delta} \sum_{j=i}^{\Delta} j n_j = \sum_{j=1}^{\Delta} j^2 n_j = \|z\|^2,$$

and the assertion follows. \square

Proof of Lemma 4. Without loss of generality we assume that $z = (z_1, \dots, z_n)$ with $z_1 \geq \dots \geq z_n \geq 0$. Let $k \in [1, n]$ be the smallest integer with $z_1^2 + \dots + z_k^2 \geq z_{k+1}^2 + \dots + z_n^2$, as in the definition of the quantity $\rho(z)$. Writing $\Delta := \lfloor \rho(z) \rfloor$ and applying Lemma 3 to the vector $z' := (\lfloor z_1/z_k \rfloor, \dots, \lfloor z_n/z_k \rfloor) \in [0, \Delta]^n$, we find a non-zero $\delta \in \{0, 1\}^n$ so that

$$\langle z', \delta \rangle \geq \frac{1}{\sqrt{\log \Delta + 1}} \|z'\| \|\delta\|.$$

It remains to notice that $\Delta \leq \rho(z)$, $\langle z', \delta \rangle \leq \langle z, \delta \rangle / z_k$, and

$$\|z'\|^2 = \sum_{i=1}^k \left\lfloor \frac{z_i}{z_k} \right\rfloor^2 > \frac{1}{4z_k^2} \sum_{i=1}^k z_i^2 \geq \frac{\|z\|^2}{8z_k^2}.$$

□

5. GRAPHS WITH $M(G) = o(\lambda_{\max}(G))$

Our goal in this section is to construct graphs G of arbitrarily large order with the largest eigenvalue $\lambda_{\max}(G)$ exceeding considerably the maximum bi-average degree $M(G)$, cf. (6). This will show that Theorems 1–3' are reasonably sharp.

The idea behind our construction is to take G to be a graph whose adjacency matrix A has a large spectral gap, and has its Perron-Frobenius eigenvector, say e , highly non-aligned with any $(0, 1)$ -vector. The former property ensures that for any vector δ , the norm $\|A\delta\|$ is controlled by the projection of δ onto e , and then the latter property shows that whenever δ is a $(0, 1)$ -vector, $\|A\delta\|$ is small. This results in $M(G)$ being small. In practice, we take A to be a high tensor power of a matrix with a large spectral gap. The spectral gap of the original matrix is then inherited by A , whereas the property of being non-aligned with $(0, 1)$ -vectors, somewhat unexpectedly, is acquired by passing to tensor powers.

Let H denote the entropy function extended by continuity onto the interval $[0, 1]$; thus, $H(x) = -x \log x - (1-x) \log(1-x)$ for $x \in (0, 1)$, and $H(0) = H(1) = 0$.

The following estimates are easy to derive using the Stirling formula:

$$\frac{1}{3} \frac{1}{\sqrt{q}} e^{tH(q/t)} < \binom{t}{q} < \frac{2}{3} \frac{1}{\sqrt{q}} e^{tH(q/t)}, \quad 1 \leq q \leq t/2. \quad (9)$$

We also need the following large deviation inequality.

Lemma 5. *For any real $\lambda > 0$ and positive integer q and t with $q \leq t/(\lambda + 1)$, we have*

$$\sum_{j=0}^q \binom{t}{j} \lambda^{t-j} \leq \lambda^{t-q} e^{tH(q/t)}.$$

Proof. Dividing through both sides of the inequality by λ^{t-q} , we get an increasing function of λ in the left-hand side and a quantity, independent of λ , in the right-hand side. Therefore, the general case will follow from that where $q = t/(\lambda + 1)$, which we now assume to hold. For brevity we write $p := q/t$, so that $\lambda + 1 = p^{-1}$ and $\frac{1-p}{p} = \lambda$. The left-hand side of the inequality in question can now be estimated from above by

$$(\lambda + 1)^t = p^{-t} = \lambda^{(1-p)t} p^{-pt} (1-p)^{-(1-p)t} = \lambda^{t-q} e^{tH(q/t)},$$

as wanted. \square

We remark that, despite its seemingly vacuous proof, the estimate of Lemma 5 is surprisingly sharp: say, numerical computations suggest that for any q, t , and λ , the right-hand side of the inequality of the lemma is at most twice larger than its left-hand side.

The reader is urged to compare our next lemma against Lemmas 2–4.

Lemma 6. *For real $\lambda \geq 4$ and integer $s, t \geq 1$, write $n = (2s)^t$, and suppose that $z \in \mathbb{R}^n$ is a vector with $\binom{t}{j} s^t$ coordinates equal to λ^j for each $j \in [0, t]$. Then for every $\delta \in \{0, 1\}^n$ we have*

$$\langle z, \delta \rangle \leq \frac{4\lambda}{\sqrt[4]{t}} \|z\| \|\delta\|.$$

Proof. Observing that $\|\delta\|^2$ is the number of coordinates of δ , equal to 1, and writing

$$\|\delta\|^2 = \sum_{j=0}^q \binom{t}{j} s^t + r,$$

we have to show that

$$\sum_{j=0}^q \binom{t}{j} s^t \lambda^{t-j} + r \lambda^{t-q-1} \leq \frac{4\lambda}{\sqrt[4]{t}} \|z\| \sqrt{\sum_{j=0}^q \binom{t}{j} s^t + r}$$

for all $0 \leq q < t$ and $0 \leq r \leq \binom{t}{q+1} s^t$. For a suitable choice of K_1, K_2 , and K_3 (depending on λ, s, t , and q), this inequality can be re-written as

$$\frac{r + K_1}{\sqrt{r + K_2}} \leq K_3.$$

Denoting the left-hand side by $f(r)$, we have

$$f'(r) = \frac{r + (2K_2 - K_1)}{2(r + K_2)^{3/2}}.$$

Consequently, either $f(r)$ is monotonic on any given closed interval, or it is decreasing on some initial segment of the interval and then increasing on the remaining segment. In any case, the maximum value of f on the interval is attained at one of its endpoints.

Hence, without loss of generality, we can focus on the case where $r = 0$ or $r = \binom{t}{q+1}s^t$; in other words, it suffices to prove that

$$\sum_{j=0}^q \binom{t}{j} s^t \lambda^{t-j} \leq \frac{4\lambda}{\sqrt[4]{t}} \|z\| \sqrt{\sum_{j=0}^q \binom{t}{j} s^t}; \quad 0 \leq q \leq t.$$

Observing that $\|z\|^2 = s^t(\lambda^2 + 1)^t$ and setting

$$S_q := \sum_{j=0}^q \binom{t}{j} \lambda^{t-j} \text{ and } \sigma_q := \sum_{j=0}^q \binom{t}{j},$$

we further rewrite the inequality to be proved as

$$S_q \leq \frac{4\lambda}{\sqrt[4]{t}} (\lambda^2 + 1)^{t/2} \sqrt{\sigma_q}; \quad 0 \leq q \leq t. \quad (10)$$

Since $S_0 = \lambda^t$ and $\sigma_0 = 1$, we have

$$\begin{aligned} \left(\frac{4\lambda}{\sqrt[4]{t}} (\lambda^2 + 1)^{t/2} \sqrt{\sigma_0} S_0^{-1} \right)^2 &\geq \frac{16\lambda^2}{\sqrt{t}} (1 + \lambda^{-2})^t \\ &\geq \frac{16\lambda^2}{\sqrt{t}} (1 + t\lambda^{-2}) = 16\lambda (\lambda t^{-1/2} + \lambda^{-1} t^{1/2}) \geq 32\lambda > 1. \end{aligned}$$

This establishes the case where $q = 0$, and we assume below that $q \geq 1$.

Let $\kappa := q/t$. We proceed by cases, splitting the interval $(0, 1]$ as

$$(0, \frac{1}{2\lambda}] \cup [\frac{1}{2\lambda}, \frac{1}{\lambda+1}] \cup [\frac{1}{\lambda+1}, \frac{1}{2}] \cup [\frac{1}{2}, 1]$$

and considering the subinterval into which κ falls.

1) Suppose first that

$$0 < \kappa \leq \frac{1}{2\lambda}. \quad (11)$$

In this case, for each $j \leq q$ we have

$$\frac{\binom{t}{j-1} \lambda^{t-(j-1)}}{\binom{t}{j} \lambda^{t-j}} = \frac{\lambda j}{t-j+1} < \frac{\lambda}{\kappa^{-1}-1} \leq \frac{\lambda}{2\lambda-1} = 1 - \eta^{-1},$$

where

$$\eta = \frac{2\lambda-1}{\lambda-1}. \quad (12)$$

Consequently,

$$S_q \leq \binom{t}{q} \lambda^{t-q} \sum_{i=0}^{\infty} (1 - \eta^{-1})^i = \eta \binom{t}{q} \lambda^{t-q},$$

and since $\sigma_q \geq \binom{t}{q}$, it suffices to show that

$$\eta^2 \binom{t}{q} \lambda^{2t-2q} \leq \frac{16\lambda^2}{\sqrt{t}} (\lambda^2 + 1)^t.$$

Using (9) and observing that $\eta < 4$ (as it follows from (12) and the assumption $\lambda \geq 4$), this can be further reduced to

$$\frac{1}{\sqrt{\kappa}} e^{tH(\kappa)} \lambda^{2t(1-\kappa)} \leq \lambda^2 (\lambda^2 + 1)^t,$$

and, by passing to logarithms, dividing through by t , and rearranging the terms, to

$$\log(\lambda^2 + 1) - 2(1 - \kappa) \log \lambda \geq H(\kappa) - \frac{1}{2t} \log(\lambda^4 \kappa).$$

Optimizing by λ , it is not difficult to see that $\log(\lambda^2 + 1) - 2(1 - \kappa) \log \lambda \geq H(\kappa)$ for all $\lambda > 0$. This settles the case where $\lambda^4 \kappa \geq 1$, and it remains to consider the situation where $\kappa \leq \lambda^{-4}$. Since $\kappa = q/t \geq t^{-1}$, it suffices to prove that in this case

$$\log(\lambda^2 + 1) - 2(1 - \kappa) \log \lambda \geq H(\kappa) - \frac{1}{2} \kappa \log(\lambda^4 \kappa);$$

equivalently,

$$\log(\lambda^2 + 1) - 2(1 - 2\kappa) \log \lambda \geq H(\kappa) - \frac{1}{2} \kappa \log(\kappa). \quad (13)$$

Since $\log(\lambda^2 + 1) - 2(1 - 2\kappa) \log \lambda$ is a decreasing function of λ in the range $4 \leq \lambda \leq \kappa^{-1/4}$, its minimum value in this range is

$$\log(\kappa^{-1/2} + 1) + \frac{1}{2} \log \kappa - \kappa \log \kappa.$$

Hence (13) will follow from

$$\log(\kappa^{-1/2} + 1) + \frac{1}{2} \log \kappa - \kappa \log \kappa \geq H(\kappa) - \frac{1}{2} \kappa \log(\kappa),$$

which simplifies to

$$\log(1 + \kappa^{1/2}) + (1 - \kappa) \log(1 - \kappa) + \frac{1}{2} \kappa \log \kappa \geq 0,$$

and in this form immediately follows from the fact that the left-hand side is an increasing function of κ on the interval $\kappa \in (0, 1/8]$; hence on the interval (11).

2) Next, suppose that

$$\frac{1}{2\lambda} \leq \kappa \leq \frac{1}{\lambda + 1} \quad (14)$$

and, as a result,

$$\lambda \geq (2\kappa)^{-1} \geq \sqrt{\kappa^{-1} - 1}. \quad (15)$$

By Lemma 5 and (9), we have

$$S_q \leq \lambda^{(1-\kappa)t} e^{tH(\kappa)} \text{ and } \sigma_q \geq \binom{t}{q} \geq \frac{1}{3\sqrt{\kappa t}} e^{tH(\kappa)}.$$

Thus, in view of (15), the result will follow from

$$\lambda^{(1-\kappa)t} e^{tH(\kappa)} \leq \frac{\kappa^{-5/4}}{\sqrt{t}} (\lambda^2 + 1)^{t/2} e^{\frac{1}{2}tH(\kappa)}.$$

By passing to logarithms, dividing through by $t/2$, and rearranging the terms, this reduces to

$$\log(\lambda^2 + 1) - 2(1 - \kappa) \log \lambda \geq H(\kappa) + t^{-1} \log(\kappa^{5/2} t).$$

Since the expression in the left-hand side is an increasing function of λ in the range $\lambda \geq \sqrt{\kappa^{-1} - 1}$, using (15) we get

$$\begin{aligned} \log(\lambda^2 + 1) - 2(1 - \kappa) \log \lambda &\geq \log(1 + 4\kappa^2) - 2\log(2\kappa) + 2(1 - \kappa) \log(2\kappa) \\ &= \log(1 + 4\kappa^2) - 2\kappa \log(2\kappa). \end{aligned}$$

Also,

$$t^{-1} \log(\kappa^{5/2} t) \leq \frac{1}{e} \kappa^{5/2}$$

for any $t > 0$. Consequently, it suffices to show that

$$\log(1 + 4\kappa^2) - 2\kappa \log(2\kappa) \geq H(\kappa) + \frac{1}{e} \kappa^{5/2},$$

and a routine investigation confirms that this holds true for all $\kappa \leq 1/5$, and therefore for all κ in the range (14).

3) Next, suppose that

$$\frac{1}{\lambda + 1} \leq \kappa \leq \frac{1}{2}. \quad (16)$$

Using the trivial estimates $S_q \leq (\lambda + 1)^t$ and $\sigma_q \geq \binom{t}{q}$, the latter in conjunction with (9), in this case we reduce (10) to

$$(\lambda + 1)^t \leq \frac{2\lambda}{\sqrt{t} \sqrt[4]{\kappa}} (\lambda^2 + 1)^{t/2} e^{\frac{1}{2}tH(\kappa)},$$

and further to

$$2\log(\lambda + 1) - \log(\lambda^2 + 1) \leq H(\kappa) - \frac{1}{t} \log(\kappa^{1/2} t / (4\lambda^2)). \quad (17)$$

By (16) we have $\lambda \geq \max\{\kappa^{-1} - 1, 4\}$, and in this range the expression in the left-hand side of (17) is easily seen to be a decreasing function of λ . As a result, we have

$$\begin{aligned} 2\log(\lambda + 1) - \log(\lambda^2 + 1) &\leq \log \frac{(1 + (\kappa^{-1} - 1))^2}{1 + (\kappa^{-1} - 1)^2} \\ &= -\log(2\kappa^2 - 2\kappa + 1) \quad \text{if } \kappa \leq \frac{1}{5}, \end{aligned}$$

and

$$2\log(\lambda + 1) - \log(\lambda^2 + 1) \leq \log \frac{25}{17} \quad \text{if } \kappa \geq \frac{1}{5}.$$

Since, on the other hand,

$$\frac{1}{t} \log(\kappa^{1/2} t / (4\lambda^2)) \leq \frac{1}{4\lambda^2 e} \kappa^{1/2} \leq \frac{1}{64e} \kappa^{1/2},$$

it suffices to show that

$$H(\kappa) - \frac{1}{64e} \kappa^{1/2} \geq \begin{cases} -\log(2\kappa^2 - 2\kappa + 1) & \text{if } \kappa \leq \frac{1}{5}, \\ \log \frac{25}{17} & \text{if } \frac{1}{5} \leq \kappa \leq \frac{1}{2}. \end{cases}$$

Again, this can be verified by a straightforward computation.

4) Finally, suppose that $\kappa \geq 1/2$. In this case we have $\sigma_q \geq 2^{t-1}$; hence, in view of $S_q \leq (\lambda + 1)^t$, it suffices to show that

$$(\lambda + 1)^t \leq \frac{1}{\sqrt[4]{t}} (2(\lambda^2 + 1))^{t/2}.$$

This can be equivalently rewritten as

$$\log(2(\lambda^2 + 1)) - 2\log(\lambda + 1) \geq \frac{1}{2t} \log t,$$

and the last inequality is immediate from the fact that its right-hand side does not exceed $1/(2e)$, while the left-hand side is an increasing function of λ in the range $\lambda \geq 4$, and its value at $\lambda = 4$ is $\log(34/25) > 1/(2e)$. \square

We can now complete our construction of graphs G with $\mathbf{M}(G)$ small (as compared to $\lambda_{\max}(G)$).

For integer $s, t \geq 1$, denote by I_s the identity matrix, and by J_s the all-1 matrix of order s , and let

$$A_s^{(t)} := \begin{pmatrix} J_s - I_s & I_s \\ I_s & 0 \end{pmatrix}^{\otimes t};$$

thus, $A_s^{(t)}$ is a symmetric $(0, 1)$ -matrix of order $(2s)^t$, with zeroes on the main diagonal.

It is not difficult to check that the minimal polynomial of $A_s^{(1)}$ is $(x^2 - (s-1)x - 1)(x^2 + x - 1)$. Letting $\lambda := (s-1 + \sqrt{(s-1)^2 + 4})/2$ (the largest root of the polynomial $x^2 - (s-1)x - 1$), we conclude that the largest eigenvalue of $A_s^{(1)}$ is equal to λ , while

all other eigenvalues do not exceed $(1 + \sqrt{5})/2$ in absolute value. Also, it is readily verified that the eigenvector corresponding to λ is $e := (\lambda, \dots, \lambda, 1, \dots, 1)$, with the $2s$ coordinates split evenly between the values λ and 1.

Write $n := (2s)^t$, and let $\lambda_1, \dots, \lambda_n$ denote the eigenvalues of $A_s^{(t)}$, with λ_1 being the largest eigenvalue. Fix an orthonormal basis $\{e_1, \dots, e_n\}$ of \mathbb{R}^n such that e_i is an eigenvector, corresponding to the eigenvalue λ_i . Since $A_s^{(t)}$ is the t th tensor power of $A_s^{(1)}$, we have $\lambda_1 = \lambda^t$, and $|\lambda_i| \leq \lambda^{t-1}(1 + \sqrt{5})/2$ for $2 \leq i \leq n$. Consequently, for any $\delta \in \mathbb{R}^n$,

$$\begin{aligned} \|A_s^{(t)}\delta\|^2 &= \lambda_1^2 \langle \delta, e_1 \rangle^2 + \lambda_2^2 \langle \delta, e_2 \rangle^2 + \dots + \lambda_n^2 \langle \delta, e_n \rangle^2 \\ &\ll \lambda^{2t} \langle \delta, e_1 \rangle^2 + \lambda^{2(t-1)} (\langle \delta, e_2 \rangle^2 + \dots + \langle \delta, e_n \rangle^2) \\ &\ll \lambda^{2t} \langle \delta, e_1 \rangle^2 + \frac{1}{s^2} \lambda^{2t} \|\delta\|^2. \end{aligned}$$

Since e_1 is proportional to the vector $e^{\otimes t}$ having $\binom{t}{j} s^t$ coordinates equal to λ^j for each $j \in [0, t]$, by Lemma 6 for any $\delta \in \{0, 1\}^n$ we have

$$\langle \delta, e_1 \rangle \leq \frac{4\lambda}{\sqrt[4]{t}} \|\delta\|;$$

as a result,

$$\|A_s^{(t)}\delta\| \ll \lambda^t \left(\frac{\lambda}{\sqrt[4]{t}} + \frac{1}{s} \right) \|\delta\|.$$

Observing that $\|A_s^{(t)}\| = \|A_s^{(1)}\|^t = \lambda^t$ and choosing $t = s^8$ to optimize, we get

$$\|A_s^{(t)}\delta\| \ll \left(\frac{\log \log n}{\log n} \right)^{1/8} \|A_s^{(t)}\| \|\delta\|, \quad \delta \in \{0, 1\}^n$$

(with an absolute implicit constant).

If we now define G to be the graph of order n with the adjacency matrix $A_s^{(t)}$, then by (3),

$$\begin{aligned} \mathbf{M}(G) &= \max_{0 \neq \xi, \eta \in \{0, 1\}^n} \frac{\langle \xi, A_s^{(t)} \eta \rangle}{\|\xi\| \|\eta\|} \\ &\leq \max_{0 \neq \eta \in \{0, 1\}^n} \frac{\|A_s^{(t)} \eta\|}{\|\eta\|} \\ &\ll \left(\frac{\log \log n}{\log n} \right)^{1/8} \|A_s^{(t)}\| \\ &= \left(\frac{\log \log n}{\log n} \right)^{1/8} \lambda_{\max}(G), \end{aligned}$$

as wanted.